XIV. *A Method of finding the Value of an infinite Series of decreasing Quantities of a certain Form, when it converges too slowly to be summed in the common Way by the mere Computation and Addition or Subtraction of some of its initial Terms.* By Francis Maseres, *Esquire,* F. R. S. *Curfitor Baron of the Exchequer.*

ARTICLE Ift. Let a, b, c, d, e, f, g, h, &c. *ad infinitum,* reprefent a decreafing progreffion of numbers, fo that b fhall be lefs than a, and c than b, and d than c, and fo on of the following numbers, *ad infinitum.*

And 2dly, let thefe numbers be fo related to each other, that they not only fhall form a decreafing progreffion themfelves, but that their differences, $a-b, b-c, c-d, d-e, e-f, f-g, g-h$, &c. fhall alfo form a decreafing progreffion, fo that $b-c$ fhall be lefs than $a-b$, and $c-d$ than $b-c$, and $d-e$ than $c-d$, and fo on of the following differences; and likewife, that the differences of thefe differences (which may be called *the fecond differences* of the original numbers a, b, c, d, e, f, g, h, &c. fhall form a decreafing progreffion; and that the differences of thofe fecond differences, or *the third differences* of the original

B b 2   numbers

numbers *a, b, c, d, e, f, g, h,* &c. fhall alſo form a de-
creaſing progreſſion; and in like manner, that the dif-
ferences of the ſaid third differences, or *the fourth dif-
ferences,* of the original numbers *a, b, c, d, e, f, g, h,* &c.
and the fifth and ſixth differences, and all higher differ-
ences, of the ſame numbers, fhall alſo form decreaſing
progreſſions.

And 3dly, let *x* be a quantity of any magnitude not
greater than unity.

Upon theſe ſuppoſitions the value of the infinite ſeries
$a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 +$ &c. (in which the
ſecond, fourth, ſixth, and eighth, and every following
even term, is marked with the ſign $-$, or is to be ſub-
tracted from that which immediately precedes it) may be
determined in the following manner.

Art. 2. Compute the firſt, ſecond, third, fourth,
and other ſubſequent differences of the co-efficients of
the powers of *x* in this ſeries, that is, of the numbers
b, c, d, e, f, g, h, &c. as far as fhall be convenient. Theſe
differences will be as follows.

Firſt differences, $b-c, c-d, d-e, e-f, f-g, g-h,$ &c.

Second differences,

$b-c-\overline{c-d}, c-d-\overline{d-e}, d-e-\overline{e-f}, e-f-\overline{f-g}, f-g-\overline{g-h},$ &c.

or, $b-2c+d, c-2d+e, d-2e+f, e-2f+g, f-2g+h,$ &c.

Third differences,

$b-2c+d-\overline{c-2d+e}, c-2d+e-\overline{d-2e+f}, d-2e+f-\overline{e-2f+g},$
$e-2f+g-\overline{f-2g+h},$ &c.

or,

or, $b-3c+3d-e$, $c-3d+3e-f$, $d-3e+3f-g$, $e-3f+3g-h$, &c.

Fourth differences, $b-3c+3d-e-\overline{c-3d+3e-f}$,

$$c-3d+3e-f-\overline{d-3e+3f-g},$$

$$d-3e+3f-g-\overline{e-3f+3g-h}, \&c.$$

or, $b-4c+6d-4e+f$, $c-4d+6e-4f+g$, $d-4e+6f-4g+h$, &c.

Fifth differences, $b-4c+6d-4e+f-\overline{c-4d+6e-4f+g}$,

$$c-4d+6e-4f+g-\overline{d-4e+6f-4g+h}, \&c.$$

or, $b-5c+10d-10e+5f-g$, $c-5d+10e-10f+5g-h$, &c.

Sixth differences,

$$b-5c+10d-10e+5f-g-\overline{c-5d+10e-10f+5g-h}, \&c.$$

or, $b-6c+15d-20e+15f-6g+h$, &c.

Let the firſt difference of the firſt order, to wit,
$b-c$, be called D^I;

and the firſt difference of the ſecond order, to wit,
$b-2c+d$, be called D^{II};

and the firſt difference of the third order, to wit,
$b-3c+3d-e$, be called D^{III};

and the firſt difference of the fourth order, to wit,
$b-4c+6d-4e+f$, be called D^{IV};

and the firſt difference of the fifth order, to wit,
$b-5c+10d-10e+5f-g$, be called D^V;

and the firſt difference of the ſixth order, to wit,
$b-6c+15d-20e+15f-6g+h$, be called D^{VI};

and in like manner let the firſt differences of the ſeventh, eighth, ninth, and tenth, and every following order of differences be denoted by D^{VII}, D^{VIII}, D^{IX}, D^X, &c. that is, by the capital letter D, with a Roman numeral figure annexed

annexed to it, expreffing the order of differences to which it belongs.

Thefe things being fuppofed, the aforefaid infinite feries $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - bx^7 +$ &c. will be equal to the following differential feries, to wit,

$$a - \frac{bx}{1+x} - \frac{D^{I}xx}{1+x|^{2}} - \frac{D^{II}x^3}{1+x|^3} - \frac{D^{III}x^4}{1+x|^4} - \frac{D^{IV}x^5}{1+x|^5} - \frac{D^{V}x^6}{1+x|^6} - \frac{D^{VI}x^7}{1+x|^7} - \&c.;$$

in which feries all the terms after the firft term a are marked with the fign $-$, or are to be fubtracted from that term.

Art. 3. If we infert the differences themfelves inftead of D^{I}, D^{II}, D^{III}, D^{IV}, D^{V}, &c. in the foregoing differential feries (which it may perhaps fometimes be convenient to do) that feries will be as follows: $a - \frac{bx}{1+x} - \overline{b-c} \times \frac{xx}{1+x|^2}$

$$- \overline{b - 2c + d} \times \frac{x^3}{1+x|^3} - \overline{b - 3c + 3d - e} \times \frac{x^4}{1+x|^4}$$

$$- \overline{b - 4c + 6d - 4e + f} \times \frac{x^5}{1+x|^5}$$

$$- \overline{b - 5c + 10d - 10e + 5f - g} \times \frac{x^6}{1+x|^6}$$

$$- \overline{b - 6c + 15d - 20e + 15f - 6g + b} \times \frac{x^7}{1+x|^7} - \&c. \text{ ad infini-}$$

tum.

Of the convergency of the foregoing differential feries.

Art. 4. The foregoing differential feries will always converge with a confiderable degree of fwiftnefs, fo

that

that fix or eight of its terms will give the value of the whole (and confequently that of the original feries $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 +$ &c. to which it is equal) exact to feveral places of figures, even in the moft difficult cafes: for if x is $= 1$ (which is its greateft poffible magnitude) $1 + x$ will be $= 1 + 1$ or 2, and confe-quently $\overline{1+x}|^2$, $\overline{1+x}|^3$, $\overline{1+x}|^4$, $\overline{1+x}|^5$, and the following powers of $1 + x$, will be equal to $4, 8, 16, 32$, and the following powers of 2; and the powers of the fraction $\frac{x}{1+x}$ will be equal to the powers of $\frac{1}{2}$. Therefore the feries

$$a - \frac{bx}{1+x} - \frac{D^I xx}{\overline{1+x}|} - \frac{D^{II} x^3}{\overline{1+x}|^3} - \frac{D^{III} x^4}{\overline{1+x}|^4} - \frac{D^{IV} x^5}{\overline{1+x}|^5} - \frac{D^V x^6}{\overline{1+x}|^6} - \frac{D^{VI} x^7}{\overline{1+x}|^7} - \&c.$$ will

in this cafe be $=$ to $a - \frac{b}{2} - \frac{D^I}{4} - \frac{D^{II}}{8} - \frac{D^{III}}{16} - \frac{D^{IV}}{32} - \frac{D^V}{64} - \frac{D^{VI}}{128} - \&c.$ the terms of which decreafe in a greater proportion than that of 1 to 2, becaufe the numerators a, b, D^I, D^{II}, D^{III}, D^{IV}, D^V, D^{VI}, &c. form a decreafing progreffion, and the denominators increafe in the proportion of 2 to 1.

Of the invefligation of the foregoing differential feries.

Art. 5. The foregoing differential feries was invefti-gated by firft, fuppofing the original feries $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 +$ &c. to be equal to another feries whofe terms fhould involve the fame powers of x as the former, but in which every power of x fhould be multiplied

multiplied into the fame power of the fraction $\frac{1}{1+x}$, in order to accelerate their convergency, and then inquiring what would be the co-efficients of the terms of fuch a feries, if fuch a feries is poffible, and what would be the figns to be prefixed to them, or in what manner they would be connected with the firft term, whether by addition or fubtraction. In order to this inquiry, I denoted the unknown co-efficients of the affumed feries by the capital letters P, Q, R, S, T, V, &c. and wrote down the terms of it near each other, without prefixing to them either of the figns + and −, but feparated them from each other only by a comma; fo that the fundamental equation, from which I derived the differential feries above-mentioned, was as follows: $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 +$ &c. is = P, $\frac{Qx}{1+x}$, $\frac{Rxx}{\overline{1+x}|^2}$, $\frac{Sx^3}{\overline{1+x}|^3}$, $\frac{Tx^4}{\overline{1+x}|^4}$, $\frac{Vx^5}{\overline{1+x}|^5}$, &c. By neceffary deductions from this equation it appeared that P would be equal to a; and that all the following terms, of the affumed feries, to wit, $\frac{Qx}{1+x}$, $\frac{Rxx}{\overline{1+x}|^2}$, $\frac{Sx^3}{\overline{1+x}|^3}$, $\frac{Tx^4}{\overline{1+x}|^4}$, $\frac{Vx^5}{\overline{1+x}|^5}$, &c. muft be fubtracted from the firft term P, or a; and that Q would be equal to $b - c$, or D^I; and $R = b - c - \overline{c - d}$, or $b - 2c + d$, or D^{II}; and $S = b - 2c + d - \overline{c - 2d + e}$, or $b - 3c + 3d - e$, or D^{III}; and

T =

$\tau \leftrightharpoons b - 4c + 6d - 4e + f$, or D^{IV}; and $v = b - 5c + 10d - 10e$ $+ 5f - g$, or D^{V}; and fo on of the following co-efficients, to wit, that every new co-efficient of the affumed feries is equal to the firft difference of the next order of the differences derived from the original co-efficients b, c, d, e, f, g, h, &c. And from hence I concluded that the feries $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + $ &c. was equal to the feries,

$$a - \frac{bx}{1+x} - \frac{\mathrm{D}^{\mathrm{I}}xx}{\overline{1+x}|^2} - \frac{\mathrm{D}^{\mathrm{II}}x^3}{\overline{1+x}|^3} - \frac{\mathrm{D}^{\mathrm{III}}x^4}{\overline{1+x}|^4} - \frac{\mathrm{D}^{\mathrm{IV}}x^5}{\overline{1+x}|^5} - \frac{\mathrm{D}^{\mathrm{V}}x^6}{\overline{1+x}|^6} - \frac{\mathrm{D}^{\mathrm{VI}}x^7}{\overline{1+x}|^7} - \text{ &c.}$$

Art. 6. The thought of fuppofing the original feries $a - bx + cxx - dx^3 + ex^4 - fx^5 + $ &c. to be equal to the feries $\mathrm{P}, \frac{\mathrm{Q}x}{1+x}, \frac{\mathrm{R}xx}{\overline{1+x}|^2}, \frac{\mathrm{S}x^3}{\overline{1+x}|^3}, \frac{\mathrm{T}x^4}{\overline{1+x}|^4}, \frac{\mathrm{V}x^5}{\overline{1+x}|^5},$ &c. containing the powers of x multiplied into the fame powers of the fraction $\frac{1}{1+x}$ in order to accelerate their convergency, occurred to me in confequence of reading the late Mr. THOMAS SIMPSON's Mathematical Differtations, p. 62, 63. concerning the fummation of feriefes, in which he makes a fuppofition of a fimilar kind. Yet there feems to be a confiderable difference between his propofition and that which is the fubject of thefe pages; for he feems to fuppofe his quantities p, q, r, s, t, &c. (which anfwer to a, b, c, d, e, &c. in the notation made ufe of in the above feriefes) to form an increafing progreffion of terms, and accordingly fubtracts p from q, and q from r,

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and *r* from *s*, and *s* from *t*, and fo on; and he feems alfo to fuppofe the differences $q-p$, $r-q$, $s-r$, $t-s$, &c. to form an increafing progreffion, and every fubfequent order of differences to form likewife an increafing progreffion, and accordingly fubtracts $q-p$ from $r-q$, and $r-q$ from $s-r$, and $s-r$ from $t-s$, and fo on; whereas in the foregoing feries $a-bx+cxx-dx^3+ex^4-fx^5+gx^6-hx^7$ + &c. the numbers a, b, c, d, e, f, g, h, &c. are fuppofed to form a decreafing progreffion of terms, as they are moft commonly found to do in the feriefes that occur in the folution of mathematical or philofophical problems.

Examples of the ufefulnefs of the foregoing differential feries in finding the values of infinite feriefes whofe terms decreafe very flowly.

Computations of the lengths of circular arcs by means of infinite feriefes derived from their tangents.

Art. 7. It is well known, that if r be put for the radius of a circle, and t for the tangent of any arch in it that is not greater than 45°, the magnitude of the arch whofe tangent is t will be expreffed by the infinite feries

$$t-\frac{t^3}{3rr}+\frac{t^5}{5r^4}-\frac{t^7}{7r^6}+\frac{t^9}{9r^8}-\frac{t^{11}}{11r^{10}}+\frac{t^{13}}{13r^{12}}-\frac{t^{15}}{15r^{14}}+ \&c.$$ This feries converges with great fwiftnefs when the tangent is

x　　　　　　　　　　　　　　　　much

much lefs than the radius; but when the tangent is
nearly equal to the radius, it converges exceeding flowly;
and when it is quite equal to the radius, or the arch is
equal to $45°$, the decreafe of the terms is fo flow as to
make the computation of it in the common way, by com-
puting the value of its initial terms, abfolutely impracti-
cable. For Sir ISAAC NEWTON has obferved concerning
this feries in that extreme cafe (which then becomes
equal to $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \frac{r}{13} - \frac{r}{15} +$ &c.) and another
feries that is almoft as flow as this, that to exhibit its
value exact to twenty decimal places of figures, there
would be occafion for no lefs than five thoufand mil-
lions of its terms, to compute which would take up
above a thoufand years. See Sir ISAAC NEWTON's fecond
letter to Mr. OLDENBURGH, dated October 24, 1676,
in the *Commercium Epiftolicum*, p. 159. In thefe
cafes therefore it will be convenient to make ufe of
fome artifice to difcover the value of the feries

$$t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} +$$ &c.; and we

fhall find the application of the differential feries above-
mentioned to be a very proper artifice for this purpofe.

Art. 8. In order to make this application, we muft con-
fider the feries $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} +$ &c.
as being the product of the multiplication of t into the

feries

feries $1 - \dfrac{tt}{3rr} + \dfrac{t^4}{5r^4} - \dfrac{t^6}{7r^6} + \dfrac{t^8}{9r^8} - \dfrac{t^{10}}{11r^{10}} + \dfrac{t^{12}}{13r^{12}} - \dfrac{t^{14}}{15r^{14}} +$ &c. and

muft fubftitute x inftead of $\dfrac{tt}{rr}$ in the terms of this laft fe-

ries, by which means it will be converted into the feries

$1 - \dfrac{x}{3} + \dfrac{xx}{5} - \dfrac{x^3}{7} + \dfrac{x^4}{9} - \dfrac{x^5}{11} + \dfrac{x^6}{13} - \dfrac{x^7}{15} +$ &c.   This feries is of

the fame form with the original feries above-mentioned,

$a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 +$ &c. the numeral

co-efficients $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}$, &c. of the powers of x

in the former feries anfwering to the literal or general

co-efficients a, b, c, d, e, f, g, h, &c. of the fame powers in

the latter feries. And thefe numeral co-efficients evidently

form a decreafing progreffion, as the co-efficients $a, b, c,$

$d, e, f, g, h,$ &c. are fuppofed to do; and we fhall find,

upon examination, that the differences of thefe numeral

co-efficients, of the feveral fucceffive orders, alfo confti-

tute decreafing progreffions, as the feveral fucceffive or-

ders of differences of the co-efficients $a, b, c, d, e, f, g, h,$

&c. are fuppofed to do.   Confequently the feries

$1 - \dfrac{x}{3} + \dfrac{xx}{5} - \dfrac{x^3}{7} + \dfrac{x^4}{9} - \dfrac{x^5}{11} + \dfrac{x^6}{13} - \dfrac{x^7}{15} +$ &c. will be equal to the dif-

ferential feries

$a - \dfrac{bx}{1+x} - \dfrac{D^I xx}{\overline{1+x}|^2} - \dfrac{D^{II} x^3}{\overline{1+x}|^3} - \dfrac{D^{III} x^4}{\overline{1+x}|^4} - \dfrac{D^{IV} x^5}{\overline{1+x}|^5} - \dfrac{D_V x^6}{\overline{1+x}|^6} - \dfrac{D^{VI} x^7}{\overline{1+x}|^7} -$ &c. if

we fuppofe the letters $a, b, c, d, e, f, g, h,$ &c. to be

equal to the numbers $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}$, &c. and

$D^I, D^{II}, D^{III}, D^{IV}, D^V, D^{VI}$, &c. to be the firft differences of

the

the feveral orders of differences of thofe numbers, be-
ginning from the fecond term $\frac{1}{3}$. Now the values of
thefe numbers, 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{13}$, $\frac{1}{15}$, &c. and of their
differences of the feveral fucceffive orders, beginning
from the fecond term $\frac{1}{3}$, will, when expreffed in decimal
fractions, be as follows:

$$1 \text{ is } = 1.000,000,000,000;$$
$$\tfrac{1}{3} = .333,333,333,333;$$
$$\tfrac{1}{5} = .200,000,000,000;$$
$$\tfrac{1}{7} = .142,857,142,857;$$
$$\tfrac{1}{9} = .111,111,111,111;$$
$$\tfrac{1}{11} = .090,909,090,909;$$
$$\tfrac{1}{13} = .076,923,076,923;$$
$$\tfrac{1}{15} = .066,666,666,666.$$

The differences of thefe numbers, beginning from the
fecond term, .333,333,333,33, are as follows:

Firft differences.

.133,333,333,333;
.057,142,857,143;
.031,746,031,746;
.020,202,020,202;
.013,986,013,986;
.010,256,410,257;
&c.

Second differences.

.076,190,476,190;
.025,396,825,397;
.011,544,011,544;
.006,216,006,216;
.003,729,603,729;
&c.

Third

Third differences.

.050,793,650,793;
.013,852,813,853;
.005,328,005,328;
.002,486,402,487;
&c.

Fourth differences.

.036,940,836,940;
.008,524,808,525;
.002,841,602,841;
&c.

Fifth differences.

.028,416,028,415;
.005,683,205,684;
&c.

Sixth differences.

.022,732,822,731;
&c.

Therefore D^I is $= .133,333,333,333$;

$\quad\quad D^{II} \quad = .076,190,476,190$;

$\quad\quad D^{III} \quad = .050,793,650,793$;

$\quad\quad D^{IV} \quad = .036,940,836,940$;

$\quad\quad D^V \quad = .028,416,028,415$;

$\quad\quad D^{VI} \quad = .022,732,822,731.$

Therefore the series $1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \frac{x^6}{13} - \frac{x^7}{15} + \&c.$ is equal to the series

$$1 - .333,333,333,333, \times \frac{x}{1+x}$$

$$- .133,333,333,333, \times \frac{xx}{\overline{1+x}|^2}$$

$$- .076,190,476,190, \times \frac{x^3}{\overline{1+x}|^3}$$

$$- .050,793,650,793, \times \frac{x^4}{\overline{1+x}|^4}$$

$$- .036,$$

$$- .036,940,836,940, \times \frac{x^5}{\overline{1+x}|^5}$$

$$- .028,416,028,415, \times \frac{x^6}{\overline{1+x}|^6}$$

$$- .022,732,822,731, \times \frac{x^7}{\overline{1+x}|^7}$$

— &c.; and confequently the product of this latter feries into the tangent t will be equal to the product of the former feries $1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \frac{x^6}{13} - \frac{x^7}{15} +$ &c. into the fame quantity, that is, to the product of the feries $1 - \frac{tt}{3rr} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \frac{t^{10}}{11r^{10}} + \frac{t^{12}}{13r^{12}} - \frac{t^{14}}{15r^{14}} +$ &c. into the tangent t, or to the original feries $t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} +$ &c. which expreffes the magnitude of the arch of which t is the tangent.

Computation of an arch of 30 degrees.

Art. 9. Now let t be the tangent of 30°, which is $= r \times \frac{1}{\sqrt{3}}$. Then will tt be $= \frac{rr}{3}$; and $\frac{tt}{rr}$, or x, $= \frac{rr}{3rr}$, or $\frac{1}{3}$. Therefore $1 + x$ will be $= 1 + \frac{1}{3} = \frac{3}{3} + \frac{1}{3} = \frac{4}{3}$, and $\frac{x}{1+x}$ will be $= \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$. Therefore $\frac{xx}{\overline{1+x}|^2}$ will be $= \frac{1}{16}$, and $\frac{x^3}{\overline{1+x}|^3}$ $= \frac{1}{64}$, and $\frac{x^4}{\overline{1+x}|^4} = \frac{1}{256}$, and $x^5 = \frac{1}{1024}$, and $x^6 = \frac{1}{4096}$, and $x^7 = \frac{1}{16384}$. Confequently the differential feries will in this

cafe be equal to

$$1 - \cdot333,333,333,333, \times \tfrac{1}{4}$$
$$- \cdot133,333,333,333, \times \tfrac{1}{16}$$
$$- .076,190,476,190, \times \tfrac{1}{64}$$
$$- .050,793,650,793, \times \tfrac{1}{256}$$
$$- .036,940,836,940, \times \tfrac{1}{1024}$$
$$- .028,416,028,415, \times \tfrac{1}{4096}$$
$$- .022,732,822,731, \times \tfrac{1}{16384}$$

$$- \&c. = 1 - .083,333,333,333,$$
$$- .008,333,333,333,$$
$$- .001,190,476,190,$$
$$- .000,198,412,698,$$
$$- .000,036,075,036,$$
$$- .000,006,937,506,$$
$$- .000,001,387,501,$$
$$- \&c.$$

$$= 1 - .093,099,955,597, = 0.906,900,044,403.$$

Therefore the feries $1 - \dfrac{x}{3} + \dfrac{xx}{5} - \dfrac{x^3}{7} + \dfrac{x^4}{9} - \dfrac{x^5}{11} + \dfrac{x^6}{13} - \dfrac{x^7}{15} + \&c.$ or

$1 - \dfrac{tt}{3rr} + \dfrac{t^4}{5r^4} - \dfrac{t^6}{7r^6} + \dfrac{t^8}{9r^8} - \dfrac{t^{10}}{11r^{10}} + \dfrac{t^{12}}{13r^{12}} - \dfrac{t^{14}}{15r^{14}} + \&c.$ is in this cafe

$= 0.906,900,044,403,$ or (neglecting the latter figures after the fixth place of figures, becaufe we are fure they

are not exact) 0.906,900. Therefore the product of the

series $1 - \dfrac{tt}{3rr} + \dfrac{t^4}{5r^4} - \dfrac{t^6}{7r^6} + \dfrac{t^8}{9r^8} - \dfrac{t^{10}}{11r^{10}} + \dfrac{t^{12}}{13r^{12}} - \dfrac{t^{14}}{15r^{14}} +$ &c. into the

tangent t is equal to $0.906,900, \times t = 0.906,900 \times r \times \dfrac{1}{\sqrt{3}}$

$\qquad = 0.906,900, \times r \times \dfrac{1}{1.732,050,8} = 0.906,900 \times r$

$\times .577,350,2 = 0.523,598,8 \times r$; that is, the series

$t - \dfrac{t^3}{3rr} + \dfrac{t^5}{5r^5} - \dfrac{t^7}{7r^6} + \dfrac{t^9}{9r^8} - \dfrac{t^{11}}{11r^{10}} + \dfrac{t^{13}}{13r^{12}} - \dfrac{t^{15}}{15r^{14}} +$ &c. (which ex-

presses the magnitude of the arch of which t is the tan-

gent) is in this case $= 0.523,598,8 \times r$, or an arch of 30°

is equal to $0.523,598,8 \times r$.

Art. 10. This value of an arch of 30° is exact in the

six first places of figures, and errs only an unit in the se-

venth figure, which should be a 7 instead of an 8, the

more exact value of that arch being $0.523,598,775,598$,

&c.   And thus by the help of only eight terms of the

differential series

$a - \dfrac{bx}{1+x} - \dfrac{\mathrm{D^I} x x}{1+x^2} - \dfrac{\mathrm{D^{II}} x^3}{1+x^3} - \dfrac{\mathrm{D^{III}} x^4}{1+x^4} - \dfrac{\mathrm{D^{IV}} x^5}{1+x^5} - \dfrac{\mathrm{D^V} x^6}{1+x^5} - \dfrac{\mathrm{D^{VI}} x^7}{1+x^7} -$ &c. we

have obtained the value of the series

$t - \dfrac{t^3}{3rr} + \dfrac{t^5}{5r^4} - \dfrac{t^7}{7r^6} + \dfrac{t^9}{9r^8} - \dfrac{t^{11}}{11r^{10}} + \dfrac{t^{13}}{13r^{12}} - \dfrac{t^{15}}{15r^{14}} +$ &c. in the case of

an arch of 30 degrees, exact to six places of figures.

This degree of exactness is the same with that which we

should attain by computing twelve terms of the series

$t - \dfrac{t^3}{3rr} + \dfrac{t^5}{5r^4} - \dfrac{t^7}{7r^6} + \dfrac{t^9}{9r^8} - \dfrac{t^{11}}{11r^{10}} + \dfrac{t^{13}}{13r^{12}} - \dfrac{t^{15}}{15r^{14}} +$ &c. itself, as will

appear from the following calculation.

Art. 11. The feries $t - \dfrac{t^3}{3rr} + \dfrac{t^5}{5r^4} - \dfrac{t^7}{7r^6} + \dfrac{t^9}{9r^8} - \dfrac{t^{11}}{11r^{10}} + \dfrac{t^{13}}{13r^{12}}$

$- \dfrac{t^{15}}{15r^{14}} + \dfrac{t^{17}}{17r^{16}} - \dfrac{t^{19}}{19r^{18}} + \dfrac{t^{21}}{21r^{20}} - \dfrac{t^{23}}{23r^{22}} +$ &c. is $= t \times$ the feries

$1 - \dfrac{tt}{3rr} + \dfrac{t^4}{5r^4} - \dfrac{t^6}{7r^6} + \dfrac{t^8}{9r^8} - \dfrac{t^{10}}{11r^{10}} + \dfrac{t^{12}}{13r^{12}} - \dfrac{t^{14}}{15r^{14}} + \dfrac{t^{16}}{17r^{16}} - \dfrac{t^{18}}{19r^{18}} + \dfrac{t^{20}}{21r^{20}}$

$- \dfrac{t^{22}}{23r^{22}} +$ &c. $=$, in the cafe of an arch of 30°, to $r \times \dfrac{1}{\sqrt{3}}$ into

the feries $1 - \dfrac{1}{3 \times 3} + \dfrac{1}{5 \times 9} - \dfrac{1}{7 \times 27} + \dfrac{1}{9 \times 81} - \dfrac{1}{11 \times 243} + \dfrac{1}{13 \times 729}$

$- \dfrac{1}{15 \times 2187} + \dfrac{1}{17 \times 6561} - \dfrac{1}{19 \times 19683} + \dfrac{1}{21 \times 59049} - \dfrac{1}{23 \times 177,147} +$ &c.

$= r \times \dfrac{1}{\sqrt{3}} \times$ the feries $1.000,000,000,000, - \dfrac{.333,333,333,333,}{3}$

$+ \dfrac{.111,111,111,111,}{5} - \dfrac{.037,037,037,037,}{7}$

$+ \dfrac{.012,345,679,012,}{9} - \dfrac{.004,115,226,337,}{11}$

$+ \dfrac{.001,371,742,112,}{13} - \dfrac{.000,457,247,370,}{15}$

$+ \dfrac{.000,152,415,790,}{17} - \dfrac{.000,050,805,263,}{19}$

$+ \dfrac{.000,016,935,087,}{21} - \dfrac{.000,005,645,029,}{23}$

$+$ &c.

$= r \times \dfrac{1}{\sqrt{3}} \times$ the feries

$1.000,000,000,000, - .111,111,111,111,$

$+ .022,222,222,222, - .005,291,005,291,$

$+ .001,371,742,112, - .000,374,111,485,$

$+ .000,105,518,624, - .000,030,483,158,$

$+ .000,008,965,634, - .000,002,673,961,$

$+ .000,000,806,432, - .000,000,245,436,$

$+$ &c.　　I

$=$

$$= r \times \tfrac{1}{\sqrt{3}} \times \overline{\left| 1.023,709,255,024, - .116,809,630,442, \right.}$$
$$+ \&c.$$

$$= r \times \tfrac{1}{\sqrt{3}} \times .906,899,624,582,$$

$$= r \times \tfrac{1}{1.732,050,8} \times .906,899,624,582,$$

$= r \times .577,350,2 \times .906,899,624,582, =$ (if we neglect the fix latter figures of 906,899,624,582, which we know to be not exact) $r \times .577,350,2 \times 906,899,$ $= r \times .523,598,319,029,8$; of which the firft fix figures .523,598, are exact.

Computation of an arch of 45 degrees.

Art. 12. Now let the tangent t be equal to the radius r, or the arch (whofe magnitude is expreffed by the feries $t - \tfrac{t^3}{3r^2} + \tfrac{t^5}{5r^4} - \tfrac{t^7}{7r^6} + \tfrac{t^9}{9r^8} - \tfrac{t^{11}}{11r^{10}} + \tfrac{t^{13}}{13r^{12}} - \tfrac{t^{15}}{15r^{14}} + $ &c.) be an arch of $45°$. This feries will, in this cafe, become equal to $r - \tfrac{r}{3} + \tfrac{r}{5} - \tfrac{r}{7} + \tfrac{r}{9} - \tfrac{r}{11} + \tfrac{r}{13} - \tfrac{r}{15} + $ &c. of which the firft eight terms will give the value of the whole exact to only one figure, as will appear by the following computation.   Thefe terms are equal to $r \times$ the eight terms $1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13} - \tfrac{1}{15}$, that is, to $r \times$ the eight terms

$$1.000,000,000,000, - .333,333,333,333,$$
$$+ .200,000,000,000, - .142,857,142,857,$$
$$+ .111,111,111,111, - .090,909,090,909,$$
$$+ .076,923,076,923, - .066,666,666,666,$$

$=$

$$= r \times \overline{1.388,034,187,034, - .633,766,233,765,}$$

$= r \times .754,267,943,269;$ which agrees with the value of the whole series $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} +$ &c. only in the highest figure 7, the more exact value of that series being .785,398,163,397, &c. But, if we compute eight terms of the differential series which is equal to the series $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} +$ &c., we shall thereby obtain its value exact to three places of figures; which is as great a degree of exactness as would be attained by computing about five hundred terms of the series $r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \frac{r}{13} - \frac{r}{15} +$ &c. itself. The computation of the eight first terms of the said differential series is as follows.

Art. 13. Since t is in this case $= r,$ tt will be $= rr,$ and consequently $\frac{tt}{rr}$, or x, will be $= 1.$ Therefore $xx,$ $x^3,$ $x^4,$ $x^5,$ and all the other powers of x, will in this case be equal to 1, and $1 + x$ will be equal to $1 + 1$, or 2, and the powers of $1 + x$ to the powers of 2. Therefore the fraction $\frac{x}{1+x}$ and its powers will be equal in this case to the fraction $\frac{1}{2}$ and its powers. Therefore the general differential series in art. 8. to wit,

$1 - .333,333,333,333, \times \frac{x}{1+x}$

$- .133,333,333,333, \times \frac{xx}{\overline{1+x}|^2}$

$- .076,190,476,190, \times \frac{x^3}{\overline{1+x}|^3}$

$- .050,793,650,793, \times \frac{x^4}{\overline{1+x}|^4}$

$- .036,940,836,940, \times \frac{x^5}{\overline{1+x}|^5}$

$- .028,416,028,415, \times \frac{x^6}{\overline{1+x}|^6}$

$- .022,732,822,731, \times \frac{x^7}{\overline{1+x}|^7}$

− &c. will become in this cafe equal to

$1 - .333,333,333,333, \times \frac{1}{2}$

$- .133,333,333,333, \times \frac{1}{4}$

$- .076,190,476,190, \times \frac{1}{8}$

$- .050,793,650,793, \times \frac{1}{16}$

$- .036,940,836,940, \times \frac{1}{32}$

$- .028,416,028,415, \times \frac{1}{64}$

$- .022,732,822,731, \times \frac{1}{128}$

$- $ &c. $= 1 - .166,666,666,666,$

$- .033,333,333,333,$

$- .009,523,809,523,$

$$- .003,174,603,174,$$
$$- .001,154,401,154,$$
$$- .000,444,000,443,$$
$$- .000,177,600,177,$$
$$- \&c.$$

$$= 1 - .214,474,414,470, = .785,525,585,530.$$

Therefore the feries $1 - \dfrac{x}{3} + \dfrac{xx}{5} - \dfrac{x^3}{7} + \dfrac{x^4}{9} - \dfrac{x^5}{11} + \dfrac{x^6}{13} - \dfrac{x^7}{15} +$ &c. or

$1 - \dfrac{tt}{3rr} + \dfrac{t^4}{5r^4} - \dfrac{t^6}{7r^6} + \dfrac{t^8}{9r^8} - \dfrac{t^{10}}{11r^{10}} + \dfrac{t^{12}}{13r^{12}} - \dfrac{t^{14}}{15r^{14}} +$ &c. is equal to

$.785,525,585,530$; and confequently the feries

$t - \dfrac{t^3}{3rr} + \dfrac{t^5}{5r^4} - \dfrac{t^7}{7r^6} + \dfrac{t^9}{9r^8} - \dfrac{t^{11}}{11r^{10}} + \dfrac{t^{13}}{13r^{12}} - \dfrac{t^{15}}{15r^{14}} +$ &c. is in this cafe

$= t \times .785,525,585,530, = r \times .785,525,585,530$; that is, the length of an arch of $45°$, in a circle whofe radius is r, is $= r \times .785,525,585,530$; which number is true to three places of figures, the more exact value of that arch being $r \times .785,398,163,397$, &c.

Art. 14. It has been afferted in art. 12, that in order to obtain the value of the feries

$t - \dfrac{t^3}{3rr} + \dfrac{t^5}{5r^4} - \dfrac{t^7}{7r^6} + \dfrac{t^9}{9r^8} - \dfrac{t^{11}}{11r^{10}} + \dfrac{t^{13}}{13r^{12}} - \dfrac{t^{15}}{15r^{14}} +$ &c. exact to 3

places of decimal figures by the mere computation of its terms, in the cafe of an arch of $45°$, we muft compute at leaft 500 of its terms. This may be proved in the following manner. The indexes of the powers of t in that feries are the odd numbers 1, 3, 5, 7, 9, 11, 13, 15,
&c.

&c. in their natural order; to which if we add an unit, the numbers thereby produced will be the even numbers 2, 4, 6, 8, 10, 12, 14, 16, &c. in their natural order, which are the doubles of the natural numbers, 1, 2, 3, 4, 5, 6, 7, &c. Therefore the number of terms of that feries from the beginning of it to any given term in it, including the faid term, is always half the number that is produced by adding an unit to the index of t in the faid term. Thus, if we take the term $\frac{t^{11}}{11\,r^{10}}$, and add 1 to 11, which is the index of the power of t in it, the fum will be 12, the half of which is 6, which is the number of terms in the feries from the beginning of it to the term $\frac{t^{11}}{11\,r^{10}}$, including the faid term, that term being the fixth term in the feries. If therefore we take the term $\frac{t^{999}}{999\,r^{998}}$, and are defirous of knowing its place in the feries, or the number of terms from the beginning of the feries to that term inclufively, we muft add 1 to the index of the power of t in its numerator, which will increafe it to 1000; and half this fum, to wit, 500, will be the number of terms from the beginning of the feries to the term $\frac{t^{999}}{999\,r^{998}}$ inclufively; or, in other words, this term will be the 500th term of the feries. To arrive therefore at thofe terms of the feries in which the in-

dexes

dexes of the powers of t are greater than 999, or 1000, or in which the numeral co-efficients of the terms (which, by the law of this feries, are equal to 1 divided by thefe indexes) are lefs than $\frac{1}{999}$ or $\frac{1}{1000}$, it is neceffary to compute 500 of its terms. Now when t is $= r$, and confequently the literal parts of the terms of this feries do not converge at all, it is evidently neceffary to carry the computation as far as thofe terms in which the numeral co-efficients of the terms are lefs than $\frac{1}{999}$ or $\frac{1}{1000}$, in order to get the value of the feries exact to the $\frac{1}{999}$th or $\frac{1}{1000}$th part of the radius r, or to the place of thoufandths, or the third place of decimal figures. Therefore, when t is $= r$, or the arch is $= 45°$, it is neceffary to compute at leaft 500 terms of the feries

$$t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} + \&c.,$$

in order to obtain the value of it exact to three places of decimal figures, that is, to the fame degree of exactnefs to which we attained in art. 13. by computing only eight terms of the above-mentioned differential feries. $Q.E.D.$

Art. 15. But the beft way of applying the aforefaid differential feries to the inveftigation of the value of one of thefe very flow feriefes, is to compute a moderate number of the firft terms of the flow feries in the common way, and then apply the differential feries to the

3   computation

computation of its remaining terms. The advantage of this method of proceeding will be manifeft, if we apply it to the foregoing example of the feries

$$t - \frac{t^3}{3rr} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} - \frac{t^{11}}{11r^{10}} + \frac{t^{13}}{13r^{12}} - \frac{t^{15}}{15r^{14}} + \&c.$$ in the cafe of an arch of 45°.

Compute therefore the firft twelve terms of this feries in the common way. Thefe terms will be as follows:

$$t = r = r \times 1.000,000,000,000;$$

$$\frac{t^3}{3rr} = \frac{r^3}{3rr} = \frac{r}{3} = r \times .333,333,333,333;$$

$$\frac{t^5}{5r^4} = \frac{r^5}{5r^4} = \frac{r}{5} = r \times .200,000,000,000;$$

$$\frac{t^7}{7r^6} = \frac{r^7}{7r^6} = \frac{r}{7} = r \times .142,857,142,857;$$

$$\frac{t^9}{9r^8} = \frac{r^9}{9r^8} = \frac{r}{9} = r \times .111,111,111,111;$$

$$\frac{t^{11}}{11r^{10}} = \frac{r^{11}}{11r^{10}} = \frac{r}{11} = r \times .090,909,090,909;$$

$$\frac{t^{13}}{13r^{12}} = \frac{r^{13}}{13r^{12}} = \frac{r}{13} = r \times .076,923,076,923;$$

$$\frac{t^{15}}{15r^{14}} = \frac{r^{15}}{15r^{14}} = \frac{r}{15} = r \times .066,666,666,666;$$

$$\frac{t^{17}}{17r^{16}} = \frac{r^{17}}{17r^{16}} = \frac{r}{17} = r \times .058,823,529,411;$$

$$\frac{t^{19}}{19r^{18}} = \frac{r^{19}}{19r^{18}} = \frac{r}{19} = r \times .052,631,578,947;$$

$$\frac{t^{21}}{21r^{20}} = \frac{r^{21}}{21r^{20}} = \frac{r}{21} = r \times .047,619,047,619;$$

$$\frac{t^{23}}{23r^{22}} = \frac{r^{23}}{23r^{22}} = \frac{r}{23} = r \times .043,478,260,869.$$

Therefore the twelve terms $t - \dfrac{t^3}{3rr} + \dfrac{t^5}{5r^4} - \dfrac{t^7}{7r^6} + \dfrac{t^9}{9r^8} - \dfrac{t^{11}}{11r^{10}}$

$+ \dfrac{t^{13}}{13r^{12}} - \dfrac{t^{15}}{15r^{14}} + \dfrac{t^{17}}{17r^{16}} - \dfrac{t^{19}}{19r^{18}} + \dfrac{t^{21}}{21r^{20}} - \dfrac{t^{23}}{23r^{22}}$ are

$$= \begin{cases} r \times 1.000,000,000,000, -r \times .333,333,333,333, \\ +r \times .200,000,000,000, -r \times .142,857,142,857, \\ +r \times .111,111,111,111, -r \times .090,909,090,909, \\ +r \times .076,923,076,923, -r \times .066,666,666,666, \\ +r \times .058,823,529,411, -r \times .052,631,578,947, \\ +r \times .047,619,047,619, -r \times .043,478,260,869, \end{cases}$$

$= r \times 1.494,476,765,064, - r \times .729,876,073,581,$

$= r \times .764,600,691,483.$

Having thus found the value of the firſt twelve terms

of the ſeries $t - \dfrac{t^3}{3rr} + \dfrac{t^5}{5r^4} - \dfrac{t^7}{7r^6} + \dfrac{t^9}{9r^8} - \dfrac{t^{11}}{11r^{10}} +$ &c. to be

$r \times .764,600,691,483$, we muſt apply the differential ſeries to the diſcovery of the value of the remaining part of this ſeries, which is the ſeries

$$\dfrac{t^{25}}{25r^{24}} - \dfrac{t^{27}}{27r^{26}} + \dfrac{t^{29}}{29r^{28}} - \dfrac{t^{31}}{31r^{30}} + \dfrac{t^{33}}{33r^{32}} - \dfrac{t^{35}}{35r^{34}} + \dfrac{t^{37}}{37r^{36}} - \dfrac{t^{39}}{39r^{38}} + \&c. \textit{ ad in-}$$

finitum. Now this ſeries is equal to the product of $\dfrac{t^{25}}{r^{24}}$ into

the ſeries $\dfrac{1}{25} - \dfrac{tt}{27rr} + \dfrac{t^4}{29r^4} - \dfrac{t^6}{31r^6} + \dfrac{t^8}{33r^8} - \dfrac{t^{10}}{35r^{10}} + \dfrac{t^{12}}{37r^{12}} - \dfrac{t^{14}}{39r^{14}} + \&c.$ or

(putting x, as before, $= \dfrac{tt}{rr}$) to the product of $\dfrac{t^{25}}{r^{24}}$ into the

ſerie

feries $\frac{1}{25} - \frac{x}{27} + \frac{xx}{29} - \frac{x^3}{31} + \frac{x^4}{33} - \frac{x^5}{35} + \frac{x^6}{37} - \frac{x^7}{39} +$ &c. which is of the fame form with the feries $a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 +$ &c. Therefore, if we put $a = \frac{1}{25}$, $b = \frac{1}{27}$, $c = \frac{1}{29}$, $d = \frac{1}{31}$, $e = \frac{1}{33}$, $f = \frac{1}{35}$, $g = \frac{1}{37}$, $h = \frac{1}{39}$, and fo on, and compute the differential feries

$$a - \frac{bx}{1+x} - \frac{D^I xx}{\overline{1+x}|^2} - \frac{D^{II} x^3}{\overline{1+x}|^3} - \frac{D^{III} x^4}{\overline{1+x}|^4} - \frac{D^{IV} x^5}{\overline{1+x}|^5} - \frac{D^V x^6}{\overline{1+x}|^6} - \frac{D^{VI} x^7}{\overline{1+x}|^7} \quad \&c.$$

thence refulting, the number thereby obtained will be the value of the feries $\frac{1}{25} - \frac{x}{27} + \frac{xx}{29} - \frac{x^3}{31} + \frac{x^4}{33} - \frac{x^5}{35} + \frac{x^6}{37} - \frac{x^7}{39} +$ &c.

or $\frac{1}{25} - \frac{tt}{27rr} + \frac{t^4}{29r^4} - \frac{t^6}{31r^6} + \frac{t^8}{33r^8} - \frac{t^{10}}{35r^{10}} + \frac{t^{12}}{37r^{12}} - \frac{t^{14}}{39r^{14}} +$ &c. This computation is as follows:

Here a is $= \frac{1}{25} = .040,000,000,000$;

$\quad b \quad = \frac{1}{27} = .037,037,037,037$;

$\quad c \quad = \frac{1}{29} = .034,482,758,620$;

$\quad d \quad = \frac{1}{31} = .032,258,064,516$;

$\quad e \quad = \frac{1}{33} = .030,303,030,303$;

$\quad f \quad = \frac{1}{35} = .028,571,428,571$;

$\quad g \quad = \frac{1}{37} = .027,027,027,027$;

$\quad h \quad = \frac{1}{39} = .025,641,025,641$.

The

The differences of thefe numbers (beginning from the fecond number $\frac{1}{27}$, or .037,037,037,037,) are as follows:

Firſt differences.

,002,554,278,416;
.002,224,694,104;
.001,955,034,413;
.001,731,601,731;
.001,544,401,544;
.001,386,001,386.

Second differences.

.000,329,584,311;
.000,269,659,891;
.000,223,432,481;
.000,187,200,187;
.000,158,400,158.

Third differences.

.000,059,924,420;
.000,046,227,409;
.000,036,232,294;
.000,028,800,028.

Fourth differences.

.000,013,697,010;
.000,009,995,115;
.000,007,432,265.

Fifth differences.

.000,003,701,894;
.000,002,562,850.

Sixth differences.

.000,001,139,044.

Therefore D^I is $= .002,554,278,416$;

$$D^{II} = .000,329,584,311;$$
$$D^{III} = .000,059,924,420;$$
$$D^{IV} = .000,013,697,010;$$
$$D^V = .000,003,701,894;$$
$$D^{VI} = .000,001,139,044.$$

Confequently the differential feries

$$a - \frac{bx}{1+x} - \frac{D^I x x}{\overline{1+x}|^2} - \frac{D^{II} x^3}{\overline{1+x}|^3} - \frac{D^{III} x^4}{\overline{1+x}|^4} - \frac{D^{IV} x^5}{\overline{1+x}|^5} - \frac{D^V x^6}{\overline{1+x}|^6} - \frac{D^{VI} x^7}{\overline{1+x}|^7} - \&c.\ is = to$$

.040,

.040,000,000,000,

— .037,037,037,037, × $\frac{x}{1+x}$

— .002,554,278,416, × $\frac{xx}{1+x|^2}$

— .000,329,584,311, × $\frac{x^3}{1+x|^3}$

— .000,059,924,420, × $\frac{x^4}{1+x|^4}$

— .000,013,697,010, × $\frac{x^5}{1+x|^5}$

— .000,003,701,894, × $\frac{x^6}{1+x|^6}$

— .000,001,139,044, × $\frac{x^7}{1+x|^7}$

— &c.

But, since t is in this case $= r, \frac{tt}{rr}$, or x, will be $= 1$, and consequently $\frac{x}{1+x}$ will be $= \frac{1}{1+1}$ or $\frac{1}{2}$. Therefore the foregoing differential series is in this case equal to

.040,000,000,000,

— .037,037,037,037, × $\frac{1}{2}$

— .002,554,278,416, × $\frac{1}{4}$

— .000,329,584,311, × $\frac{1}{8}$

— .000,059,924,420, × $\frac{1}{16}$

— .000,013,697,010, × $\frac{1}{32}$

— .000,

$$- .000,003,701,894, \times \tfrac{1}{64}$$

$$- .000,001,139,044, \times \tfrac{1}{128}$$

$$
\begin{aligned}
- \&c. = .040,000,000,000, &- .018,518,518,518, \\
&- .000,638,569,604, \\
&- .000,041,198,038, \\
&- .000,003,745,276, \\
&- .000,000,428,031, \\
&- .000,000,057,842, \\
&- .000,000,008,898, \\
&- \&c.
\end{aligned}
$$

$$= .040,000,000,000, - .019,202,526,207, - \&c.$$

$$= .020,797,473,793 - \&c.$$

Therefore the feries $\dfrac{1}{25} - \dfrac{x}{27} + \dfrac{xx}{29} - \dfrac{x^3}{31} + \dfrac{x^4}{33} - \dfrac{x^5}{35} + \dfrac{x^6}{37} - \dfrac{x^7}{39} + \&c.$ or

$\dfrac{1}{25} - \dfrac{tt}{27rr} + \dfrac{t^4}{29r^4} - \dfrac{t^6}{31r^6} + \dfrac{t^8}{33r^8} - \dfrac{t^{10}}{35r^{10}} + \dfrac{t^{12}}{37r^{12}} - \dfrac{t^{14}}{39r^{14}} + \&c.$ is in this

cafe $= .020,797,473,793.$ Therefore the feries

$\dfrac{t^{25}}{25r^{24}} - \dfrac{t^{27}}{27r^{26}} + \dfrac{t^{29}}{29r^{28}} - \dfrac{t^{31}}{31r^{30}} + \dfrac{t^{33}}{33r^{32}} - \dfrac{t^{35}}{35r^{34}} + \dfrac{t^{37}}{37r^{36}} - \dfrac{t^{39}}{39r^{38}} + \&c.$ is in

this cafe equal to $\dfrac{t^{25}}{r^{24}} \times .020,797,473,793,$ that is, to

$\dfrac{r^{25}}{r^{24}} \times .020,797,473,793,$ or $r \times .020,797,473,793;$ that

is, the remainder of the infinite feries

$t - \dfrac{t^3}{3rr} + \dfrac{t^5}{5r^4} - \dfrac{t^7}{7r^6} + \dfrac{t^9}{9r^8} - \dfrac{t^{11}}{11r^{10}} + \&c.$ after the firft twelve

terms, is $= r \times .020,797,473,793.$ But we before found

7   thofe

thofe firft twelve terms to be $= r \times .764,600,691,483$.
Therefore the whole feries $t - \dfrac{t^3}{3rr} + \dfrac{t^5}{5r^4} - \dfrac{t^7}{7r^6} + \dfrac{t^9}{9r^8} - \dfrac{t^{11}}{11r^{10}} +$
&c. *ad infinitum* is in this cafe $= r \times .764,600,691,483$,
$+ r \times .020,797,473,793, = r \times .785,398,165,276$,
which is true to eight places of figures, the more exact
value of that feries being $r \times .785,398,163,397$, &c.;
fo that the value here found for this feries, by the help
of only eight terms of the differential feries, differs from
its true value by lefs than an unit in the eighth place of
decimal figures, that is, by lefs than an hundred-mil-
lionth part of the radius r, which is a degree of exact-
nefs that could not have been attained by the mere com-
putation of the feries $t - \dfrac{t^3}{3rr} + \dfrac{t^5}{5r^4} - \dfrac{t^7}{7r^6} + \dfrac{t^9}{9r^8} - \dfrac{t^{11}}{11r^{10}} +$ &c. it-
felf without computing fifty millions of its terms. There
cannot be a ftronger inftance of the utility of that dif-
ferential feries.

*Computation of the feries which expreffes the time of the
defcent of a pendulum through the arch of a circle.*

Art. 16. As another example of the utility of the
foregoing differential feries in finding the value of a fe-
ries that converges very flowly, I will now apply it to
the feries which expreffes the time of defcent of a
heavy body through a circular arch of 90°, which de-
creafes

creafes almoft as flowly as the above-mentioned feries

$$r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \frac{r}{11} + \frac{r}{13} - \frac{r}{15} +$$ &c., which expreffes the magnitude of a circular arch of 45° in a circle whofe radius is *r*.

Art. 17. If a heavy body, or a pendulum, be fuppofed to defcend by the mere force of gravity through any arch of a circle not exceeding the arch of a quadrant, or 90°; and the motion be fuppofed to begin from a ftate of reft, and to continue till the bob of the pendulum, or the heavy body, comes to the loweft point of the circle; and the radius of the circle be called *r*, the perpendicular height, or verfed fine, of the arch through which the defcent is made, be called *v*, and the right fine of the fame arch be called *s*; and π be put for the number 1.570,796,326,794, &c. which expreffes the femi-circumference of a circle whofe diameter is called 1; and the time of the fall of a heavy body through the verfed fine *v*, or the perpendicular altitude of the arch through which the pendulum defcends, be denoted by *v*; the time of the defcent of the pendulum through the faid circular arch, correfponding to the verfed fine or altitude *v*, to the loweft point of the circle, will be expreffed by the product of $\pi \times \dfrac{rv}{s}$ into the feries

$$1 - \frac{1.1}{2.2} \times \frac{vv}{ss} + \frac{1.1.3.3}{2.2.4.4} \times \frac{v^4}{s^4} - \frac{1.1.3.3.5.5}{2.2.4.4.6.6} \times \frac{v^6}{s^6} + \frac{1.1.3.3.5.5.7.7}{2.2.4.4.6.6.8.8} \times \frac{v^8}{s^8}$$

$$-\frac{1.1.3.3.5.5.7.7.9.9}{2.2.4.4.6.6.8.8.10.10} \times \frac{v^{10}}{s^{10}} + \&c.$$ in which the law of the continuation of the terms is very manifeſt, every new term being derived from the preceding term by multiplying it into the fraction $\frac{vv}{ss}$, and likewiſe into a numeral fraction, whoſe denominator is the ſquare of the index of the powers of v and s in the new term, and whoſe numerator is the ſquare of the odd number that is leſs than the ſaid index by an unit.

Art. 18. Let the numeral co-efficients of the terms of this ſeries be denoted by the capital letters of the alphabet, A, B, C, D, E, F, G, H, &c. in their natural order, ſo that A ſhall be equal to 1, and B ſhall be $=$ $\frac{1.1}{2.2}$, and $C = \frac{1.1.3.3}{2.2.4.4}$, and $D = \frac{1.1.3.3.5.5}{2.2.4.4.6.6}$, and ſo on of the reſt. And we ſhall have

$$B = \frac{1.1}{2.2} A, \quad C = \frac{3.3}{4.4} B, \quad D = \frac{5.5}{6.6} C, \quad E = \frac{7.7}{8.8} D, \quad F = \frac{9.9}{10.10} E, \quad G = \frac{11.11}{12.12} F, \quad H =$$

$\frac{13.13}{14.14} G$, and ſo on; and conſequently the ſeries

$$1 - \frac{1.1}{2.2} \times \frac{vv}{ss} + \frac{1.1.3.3}{2.2.4.4} \times \frac{v^4}{s^4} - \frac{1.1.3.3.5.5}{2.2.4.4.6.6} \times \frac{v^6}{s^6} + \&c.$$

or $A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} + \frac{Gv^{12}}{s^{12}} - \frac{Hv^{14}}{s^{14}} + \&c.$ will be $A -$

$$\frac{1.1.Avv}{2.2. \ ss} + \frac{3.3.Bv^4}{4.4. \ s^4} - \frac{5.5.Cv^6}{6.6. \ s^6} + \frac{7.7 \ Dv^8}{8.8. \ s^8} - \frac{9.9. \ Ev^{10}}{10.10. \ s^{10}} + \frac{11.11.Fv^{12}}{12.12. \ s^{12}} - \frac{13.13.Gv^{14}}{14 \ 14. \ s^{14}}$$

$+ \&c.$ or $A - \frac{1Avv}{4ss} + \frac{9Bv^4}{16s^4} - \frac{25Cv^6}{36s^6} + \frac{49Dv^8}{64s^8} - \frac{81Ev^{10}}{100s^{10}} + \frac{121Fv^{12}}{144s^{12}} - \frac{169Gv^{14}}{196s^{14}}$

$+ \&c.$ or, if we convert the co-efficients of the terms into decimal fractions,

$$1 - .250,000,000,000, \times \frac{vv}{ss} + .140,625,000,000, \times \frac{v^4}{s^4}$$

$$- .097,656,250,000, \times \frac{v^6}{s^6} + .074,768,066,406, \times \frac{v^8}{s^8}$$

$$- .060,562,133,788, \times \frac{v^{10}}{s^{10}} + .050,889,015,196, \times \frac{v^{12}}{s^{12}}$$

$$- .043,878,793,714, \times \frac{v^{14}}{s^{14}} + \&c.$$

The co-efficients of thefe terms decreafe fo flowly (efpecially after the firft twelve or fourteen terms) that, when the verfed fine v is very nearly equal to the right fine s (as is the cafe when the arch through which the heavy body defcends is nearly equal to 90°, or the arch of a whole quadrant of a circle) it would be neceffary to compute a vaft number of the terms of the feries in order to obtain its value exact to feven or eight places of figures; and, when v is quite equal to s (as is the cafe when the arch, through which the defcent is made, is exactly equal to 90°) the computation of the value of the feries to that degree of exact-nefs in that direct manner becomes wholly impractica-ble. But by the help of the differential feries above-mentioned its value may be found, even in this cafe, to that degree of exactnefs without much difficulty; more efpecially if we compute the firft twelve terms of the feries in the common way, and then apply the differen-tial feries to the inveftigation of the remaining part of it in the fame manner as in the laft example. This we fhall now proceed to do.

Art. 19. The co-efficients of the first twelve terms of the series

$$A - \frac{1 A\, vv}{4ss} + \frac{9 B\, v^4}{16 s^4} - \frac{25 C\, v^6}{36 s^6} + \frac{49 D\, v^8}{64 s^8} - \frac{81 E\, v^{10}}{100 s^{10}} + \frac{121 F\, v^{12}}{144 s^{12}}$$

$$- \frac{169 G\, v^{14}}{196 s^{14}} + \&c.$$ are as follows:

$$A = 1 = 1.000,000,000,000;$$

$$B = \frac{1}{4} = .250,000,000,000;$$

$$C = \frac{9 B}{16} = .140,625,000,000;$$

$$D = \frac{25 C}{36} = .097,656,250,000;$$

$$E = \frac{49 D}{64} = .074,768,066,406;$$

$$F = \frac{81 E}{100} = .060,562,133,788;$$

$$G = \frac{121 F}{144} = .050,889,015,196;$$

$$H = \frac{169 G}{196} = .043,878,793,714;$$

$$I = \frac{225 H}{256} = .038,565,346,037;$$

$$K = \frac{289 I}{324} = .034,399,336,434;$$

$$L = \frac{361 K}{400} = .031,045,401,131;$$

$$M = \frac{441 L}{484} = .028,287,235,328.$$

But when v is $= s$, as it is in the case of an arch of $90°$, $\frac{vv}{ss}$ and all its powers will be $= 1$, and the twelve terms

$$A - \frac{B\, vv}{ss} + \frac{C\, v^4}{s^4} - \frac{D\, v^6}{s^6} + \frac{E\, v^8}{s^8} - \frac{F\, v^{10}}{s^{10}} + \frac{G\, v^{12}}{s^{12}} - \frac{H\, v^{14}}{s^{14}} + \frac{I\, v^{16}}{s^{16}} - \frac{K\, v^{18}}{s^{18}} + \frac{L\, v^{20}}{s^{20}} - \frac{M\, v^{22}}{s^{22}}$$

will

will be equal to their co-efficients A−B+C−D+E−F+G−
H+I−K+L−M. Therefore in this cafe the firſt twelve
terms of this ſeries are

$$1.000,000,000,000, - .250,000,000,000,$$
$$+ \ .140,625,000,000, - .097,656,250,000,$$
$$+ \ .074,768,066,406, - .060,562,133,788,$$
$$+ \ .050,889,015,196, - .043,878,793,714,$$
$$+ \ .038,565,346,037, - .034,399,336,434,$$
$$+ \ .031,045,401,131, - .028,287,235,328,$$

which are $= 1.335,892,828,770, - .514,783,749,264,$
$$= .821,109,079,506.$$

Art. 20. The remaining part of this ſeries is

$$\frac{N v^{24}}{s^{24}} - \frac{O v^{26}}{s^{26}} + \frac{P v^{28}}{s^{28}} - \frac{Q v^{30}}{s^{30}} + \frac{R v^{32}}{s^{32}} - \frac{S v^{34}}{s^{34}} + \frac{T v^{36}}{s^{36}} - \frac{V v^{38}}{s^{38}} + \&c.$$

or $\dfrac{23.23.M v^{24}}{24.24.s^{24}} - \dfrac{25.25.N v^{26}}{26.26.s^{26}} + \dfrac{27.27.O v^{28}}{28.28.s^{28}} - \dfrac{29.29.P v^{30}}{30.30.s^{30}} + \dfrac{31.31.Q v^{32}}{32.32.s^{32}}$

$$- \frac{33.33.R v^{34}}{34.34.s^{34}} + \frac{35.35.S v^{36}}{36.36.s^{36}} - \frac{37.37.T v^{38}}{38.38.s^{38}} + \&c. \text{ or}$$

$$.025,979,075,500, \times \frac{v^{24}}{s^{24}} - .024,019,115,661 \times \frac{v^{26}}{s^{26}}$$

$$+ .022,334,101,169, \times \frac{v^{28}}{s^{28}} - .020,869,976,759 \times \frac{v^{30}}{s^{30}}$$

$$+ .019,585,984,048, \times \frac{v^{32}}{s^{32}} - .018,450,810,232, \times \frac{v^{34}}{s^{34}}$$

$$+ .017,440,001,955, \times \frac{v^{36}}{s^{36}} - .016,534,184,679, \times \frac{v^{38}}{s^{38}}$$

$+ \&c.;$ which is $= \dfrac{v^{24}}{s^{24}} \times$ the ſeries

$$.025,979,075,500, - .024,019,115,661, \frac{vv}{ss}$$

$$+ .022,$$

$$+ \ .022,334,101,169, \frac{v^4}{s^4} - .020,869,976,759, \frac{v^6}{s^6}$$

$$+ \ .019,585,984,048, \frac{v^8}{s^8} - .018,450,810,232, \frac{v^{10}}{s^{10}}$$

$$+ \ .017,440,001,955, \frac{v^{12}}{s^{12}} - .016,534,184,679, \frac{v^{14}}{s^{14}}$$

+ &c. or, if we fubftitute x in this laft feries inftead of $\frac{vv}{ss}$, $= \frac{v^{24}}{s^{24}} \times$ the feries

$$.025,979,075,500, - .024,019,115,661, x$$

$$+ \ .022,334,101,169, xx - .020,869,976,759, x^3$$

$$+ \ .019,585,984,048, x^4 - .018,450,810,232, x^5$$

$$+ \ .017,440,001,955, x^6 - .016,534,184,679, x^7$$

+ &c. Now the value of this laft feries may be difco-vered by the application of the differential feries

$$a - \frac{bx}{1+x} - \frac{\mathrm{D^I} xx}{\overline{1+x}|^2} - \frac{\mathrm{D^{II}} x^3}{\overline{1+x}|^3} - \frac{\mathrm{D^{III}} x^4}{\overline{1+x}|^4} - \frac{\mathrm{D^{IV}} x^5}{\overline{1+x}|^5} - \frac{\mathrm{D^{V}} x^6}{\overline{1+x}|^6} - \frac{\mathrm{D^{VI}} x^7}{\overline{1+x}|^7} - \ \&c.$$

in the manner following:

Here a is $= .025,979,075,500$;

$\quad b \ = .024,019,115,661$;

$\quad c \ = .022,334,101,169$;

$\quad d \ = .020,869,976,759$;

$\quad e \ = .019,585,984,048$;

$\quad f \ = .018,450,810,232$;

$\quad g \ = .017,440,001,955$;

and $\ b \ = .016,534,184,679$.

Therefore

Therefore the differences of b, c, d, e, f, g, and h, of the several succeffive orders, are as follows:

First differences.

$b - c$ is $= .001,685,014,492$;
$c - d\quad = .001,464,124,410$;
$d - e\quad = .001,283,992,711$;
$e - f\quad = .001,135,173,816$:
$f - g\quad = .001,010,808,277$;
$g - h\quad = .000,905,817,276$.

Second differences.

$.000,220,890,082$;
$.000,180,131,699$;
$.000,148,818,895$;
$.000,124,365,539$;
$.000,104,991,001$.

Third differences.

$.000,040,758,383$;
$.000,031,312,804$;
$.000,024,453,356$;
$.000,019,374,538$.

Fourth differences.

$.000,009,445,579$;
$.000,006,859,448$;
$.000,005,078,818$.

Fifth differences.

$.000,002,586,131$;
$.000,001,780,630$.

Sixth differences.

$.000,000,805,501$.

Therefore D^{I} is $= .001,685,014,492$;

$$D^{II} = .000,220,890,082;$$
$$D^{III} = .000,040,758,383;$$
$$D^{IV} = .000,009,445,579;$$
$$D^{V} = .000,002,586,131;$$
$$D^{VI} = .000,000,805,501.$$

Confequently the differential feries

$$a - \frac{bx}{1+x} - \frac{D^{I}x x}{1+x|^{2}} - \frac{D^{II}x^{3}}{1+x|^{3}} - \frac{D^{III}x^{4}}{1+x|^{4}} - \frac{D^{IV}x^{5}}{1+x|^{5}} - \frac{D^{V}x^{6}}{1+x|^{6}} - \frac{D^{VI}x^{7}}{1+x|^{7}} - \&c.\ \text{is} = \text{to}$$

$.025,$

.025,979,075,500,

$- .024,019,115,661, \times \frac{x}{1+x}$

$- .001,685,014,492, \times \frac{xx}{\overline{1+x}|^2}$

$- .000,220,890,082, \times \frac{x^3}{\overline{1+x}|^3}$

$- .000,040,758,383, \times \frac{x^4}{\overline{1+x}|^4}$

$- .000,009,445,579, \times \frac{x^5}{\overline{1+x}|^5}$

$- .000,002,586,131, \times \frac{x^6}{\overline{1+x}|^6}$

$- .000,000,805,501, \times \frac{x^7}{\overline{1+x}|^7}$

$-$ &c.

This is the general value of the faid differential feries, whatever may be the value of x, or $\frac{vv}{ss}$. But in the cafe here fuppofed of an arch of 90°, the verfed fine v is equal to the fine s; and therefore $\frac{vv}{ss}$, or x, is $= 1$, and $\frac{x}{1+x} = \frac{1}{1+1}$, or $\frac{1}{2}$. Therefore the foregoing differential feries is in this cafe equal to

.025,979,075,500,

$- .024,019,115,661, \times \frac{1}{2}$

$- .001,685,014,492, \times \frac{1}{4}$

$- .000,$

$$- .000,220,890,082, \times \tfrac{1}{8}$$

$$- .000,040,758,383, \times \tfrac{1}{16}$$

$$- .000,009,445,579, \times \tfrac{1}{32}$$

$$- .000,002,586,131, \times \tfrac{1}{64}$$

$$- .000,000,805,501, \times \tfrac{1}{128}$$

$- \&c. = .025,979,075,500 \quad - .012,009,557,830,$

$\qquad\qquad\qquad\qquad\qquad - .000,421,253,623,$

$\qquad\qquad\qquad\qquad\qquad - .000,027,611,260,$

$\qquad\qquad\qquad\qquad\qquad - .000,002,547,398,$

$\qquad\qquad\qquad\qquad\qquad - .000,000,295,174,$

$\qquad\qquad\qquad\qquad\qquad - .000,000,040,408,$

$\qquad\qquad\qquad\qquad\qquad - .000,000,006,292,$

$\qquad\qquad\qquad\qquad\qquad - \&c.$

$= .025,979,075,500, - .012,461,311,985, \&c.$

$= .013,517,763,515, - \&c.$ Therefore the feries

$a - bx + cxx - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c.$, or

$.025,979,075,500 - .024,019,115,661, x$

$+ .022,334,101,169, xx - .020,869,976,759, x^3$

$+ .019,585,984,048, x^4 - .018,450,810,232, x^5$

$+ .017,440,001,955, x^6 - .016,534,184,679, x^7$

$+ \&c.$ is in this cafe $= .013,517,763,515, - \&c.$

Therefore $\frac{v^{24}}{j^{24}} \times$ this laft feries, or $\frac{v^{24}}{j^{24}} \times$ the feries

.025

$.025,979,075,500, - .024,019,115,661, \frac{vv}{ss}$

$+ .022,334,101,169, \frac{v^4}{s^4} - .020,869,976,759, \frac{v^6}{s^6}$

$+ .019,585,984,048, \frac{v^8}{s^8} - .018,450,810,232, \frac{v^{10}}{s^{10}}$

$+ .017,440,001,955, \frac{v^{12}}{s^{12}} - .016,534,184,679, \frac{v^{14}}{s^{14}}$

$+$ &c. is in this cafe $= \frac{v^{24}}{s^{24}} \times .013,517,763,515, -$ &c.;
that is, the feries

$.025,979,075,500, \frac{v^{24}}{s^{24}} - .024,019,115,661, \frac{v^{26}}{s^{26}}$

$+ .022,334,101,169, \frac{v^{28}}{s^{28}} - .020,869,976,759, \frac{v^{30}}{s^{30}}$

$+ .019,585,984,048, \frac{v^{32}}{s^{32}} - .018,450,810,232, \times \frac{v^{34}}{s^{34}}$

$+ .017,440,001,955, \frac{v^{36}}{s^{36}} - .016,534,184,679, \frac{v^{38}}{s^{38}}$

$+$&c., or $\frac{Nv^{24}}{s^{24}} - \frac{Ov^{26}}{s^{26}} + \frac{Pv^{28}}{s^{28}} - \frac{Qv^{30}}{s^{30}} + \frac{Rv^{32}}{s^{32}} - \frac{Sv^{34}}{s^{34}} + \frac{Tv^{36}}{s^{36}} - \frac{Vv^{38}}{s^{38}} +$ &c. is
in this cafe $= \frac{v^{24}}{s^{24}} \times .013,517,763,515, -$ &c. $=$ (becaufe
v is in this cafe $= s$, and confequently $\frac{v^{24}}{s^{24}}$ is $= 1$)
$.013,517,763,515, -$ &c. But we before found the
value of the firft twelve terms of the feries
$A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} +$ &c. to be in this cafe $=$
$.821,109,079,506.$ Therefore the value of the whole
feries $A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}} +$ &c. *ad infinitum* is in
this cafe $= .821,109,079,506, + .013,517,763,515,$

– &c. = 834,626,843,021, – &c. of which the firſt eight figures .834,626,84 are exaƈt, the error being in the ninth figure 3, which ought to be a 2 inſtead of a 3, as would have appeared if we had computed another term or two of the differential ſeries.

Art. 21. Since the ſeries $A - \frac{Bvv}{ss} + \frac{Cv^4}{s^4} - \frac{Dv^6}{s^6} + \frac{Ev^8}{s^8} - \frac{Fv^{10}}{s^{10}}$

+ &c., or $1 - \frac{1.1.Avv}{2.2.ss} + \frac{3.3.Bv^4}{4.4.s^4} - \frac{5.5.Cv^6}{6.6.s^6} + \frac{7.7.Dv^8}{8.8.s^8} - \frac{9.9.Ev^{10}}{10.10.s^{10}} -$ &c.

is, in this caſe of an arch of 90°, = .834,626,843, – &c., or ſomewhat leſs than .834,626,843, the produƈt of that

ſeries into $\pi \times \frac{rv}{s}$, will be $= \pi \times \frac{rv}{s} \times .834,626,843, -$ &c. =

(becauſe v is in this caſe = s) $\pi \times r \times .834,626,843$, – &c.

= 1.570,796,326,794, &c. $\times r \times .834,626,843$

– &c. = $r \times$ 1.311,028,779, – &c., or ſomething leſs than $r \times$ 1.311,028,779; which is exaƈt to nine places of figures, the more exaƈt value of this quantity being $r \times$ 1.311,028,777,146, &c. as appears by a computation made by Mr. STIRLING, in his admirable Treatiſe on the Summation of Seriefes, p. 58.

Art. 22. This value of the produƈt of 1.570,796, 326,794, &c. $\times \frac{rv}{s}$ into the ſeries

$1 - \frac{1.1.Avv}{2.2.ss} + \frac{3.3.Bv^4}{4.4.s^4} - \frac{5.5.Cv^6}{6.6.s^6} + \frac{7.7.Dv^8}{8.8.s^8} - \frac{9.9.Ev^{10}}{10.10.s^{10}} +$ &c., found by the foregoing proceſſes in this extreme and moſt difficult caſe, to wit, 1.311,028,779, &c. $\times r$, exceeds its true

value,

value, 1.311,028,777,146, &c. × r by only .000,000, 002, × r, or two thouſand-millionth parts of the radius r; which is indeed a moſt minute difference, and ſhews the great exactneſs and utility of this differential ſeries.

Art. 23. Of the nine figures to which the number 1.311,028,779, found by the foregoing proceſs, is exact, the laſt eight are owing to the differential ſeries. For if we were to multiply the value of the firſt twelve terms only of the ſeries $A - \frac{B vv}{ss} + \frac{C v^4}{s^4} - \frac{D v^6}{s^6} + \frac{E v^8}{s^8} - \frac{F v^{10}}{s^{10}} + $ &c, or

$1 - \frac{1 \cdot 1.A vv}{2.2.ss} + \frac{3.3.B v^4}{4 4.s^4} - \frac{5.5.C v^6}{6.6.s^6} + \frac{7.7.D v^8}{8.8.s^8} - \frac{9.9 E v^{10}}{10.10.s^{10}} + $ &c., to wit, the number .821,109,079,506, into $\pi \times r$, or 1.570,796, 326,794, &c. × r, the product would be only 1.289, &c. × r, which is true to only one place of figures, the ſecond figure being a 2 inſtead of a 3. This therefore is an eminent proof of the utility of the ſaid differential ſeries.

Art. 24. In an arch of 90° the verſed ſine is equal to the radius of the circle, that is, according to the fore-going notation, v is $= r$. Therefore by art. 17. together with the foregoing computation, it appears, that the time of the deſcent of a pendulum, or other heavy body (moving freely from a ſtate of reſt by the force of gravity only) through the arch of a whole quadrant of a circle is to the time of the fall through the cor-

reſpondent

refpondent perpendicular altitude, or the radius, as 1.311,028,779, – &c. × *r* is to *r*, or as 1.311,028,779, – &c. is to 1.

Art. 25. Hence we may determine the proportion of the time of defcent of a pendulum through an arch of 90° to the time of its defcent through an infinitely fmall arch at the bottom of a quadrant, or rather (to fpeak correctly) to the limit of the time of defcent through a very fmall but finite arch at the bottom of the quadrant, to which the faid time continually approaches nearer and nearer as the faid fmall arch is taken lefs and lefs, and to which it may be made to approach fo nearly, by taking the faid fmall arch fufficiently fmall, as to differ from it by lefs than any given quantity. For this latter time, or limit, is known to be to the time of the fall of a heavy body through half the length of the pendulum, or half the radius of the circle, as the femi-circumference of a circle is to its diameter, that is, as the number 1.570,796,326,794, &c. is to 1. But the time of the fall of a heavy body through half the radius of the circle is to the time of the fall through the whole radius as 1 to √2, or 1.414,213, &c. Therefore, *ex æquo*, the faid limit of the time of defcent of a pendulum through a very fmall arch of the circle at the bottom of the qua-drant, is to the time of the fall of a heavy body through

6   the

the radius of the circle, or the whole length of the pen-
dulum, as 1.570,796, &c. is to 1.414,213, &c. But
we have feen in the laft article that the time of the fall
of a heavy body through the radius of the circle is to
the time of defcent of a pendulum through the arch of
a whole quadrant as 1 to 1.311,028,779, – &c. There-
fore the limit of the time of defcent of a pendulum
through a very fmall arch at the bottom of the quadrant
is to the time of defcent through the arch of the whole
quadrant as 1.570,796,326, &c. × 1 is to 1.414,213,
&c. × 1.311,028,779, – &c., or as 1.570,796,326, &c.
is to 1.414,213, &c. × 1.311,028,779, – &c., that is,
by art. 21. as 1.570,796,326, &c. is to 1.414,213, &c.
× 1.570,796,326, &c. × .834,626,843, – &c., or as
1 to 1.414,213, &c. × .834,626,843, – &c., or as 1 to
1.180,340, &c., or, in fmaller numbers, as 1 to 1.180,
or as 1000 to 1180, or as 100 to 118, or as 50 to 59.

Art. 26. This proportion of the times of the de-
fcent of a pendulum through an infinitely fmall arch
at the bottom of the quadrant, and through the arch
of the whole quadrant, agrees pretty nearly with
that affigned for them by Mr. HUYGENS in the pre-
face to his admirable Treatife on Pendulum-clocks,
or *De Horologio Ofcillatorio*, which is that of 29 to
34. For 50 is to 59 as 29 is to 34.2, or $34\frac{1}{5}$; or
(neglecting the fraction $\frac{1}{5}$) as 29 is to 34; Mr. HUYGENS

meaning

meaning, probably, in that place, not to exprefs this pro-
portion as accurately as he could, but only as nearly as it
could be expreffed by fmall whole numbers. However,
the numbers 50 and 59 exprefs this proportion rather
more accurately than 29 and 34, and with pretty much
the fame degree of fimplicity, and therefore, upon the
whole, are fomewhat to be preferred to them.

Art. 27. I have endeavoured to find another differen-
tial feries, fimilar to that above defcribed, for the purpofe
of inveftigating the value of an infinite feries of this
form, to wit, $a+bx+cxx+dx^3+ex^4+fx^5+gx^6+hx^7+$ &c.
(in which all the terms are marked with the fign +, or
are added to the firft term a) when the co-efficients b, c,
d, e, f, g, h, &c. decreafe very flowly, and x is very nearly
equal to 1, and the terms of the feries decreafe confe-
quently fo flowly as to make the fummation of it in the
common way, or by the mere computation and addition
of its terms, almoft impracticable; but my endeavours
have not been attended with fuccefs. I may therefore,
from my own experience, fubfcribe to the truth of what
is afferted upon this fubject by the very learned and in-
genious Mr. JAMES STIRLING in his Treatife, intitled,
Summatio Serierum, p. 17. to wit, that *Series quarum
termini funt per vices negativi et affirmativi, funt magis
tractabiles quam alteræ, ubi de Summatione agitur*;
though at firft fight one would be apt to imagine the re-
verfe of this propofition to be true.